

A Radon-type transform arising in photoacoustic tomography with circular detectors

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Abstract

Photoacoustic tomography is the most well-known example of a hybrid imaging method. In this article, we define a Radon-type transform arising in a version of photoacoustic tomography that uses integrating circular detectors and describe how the Radon transform integrating over all circles with a fixed radius is determined from this Radon-type transform. Here we consider three situations: when the centers of the circular detectors are located on a cylinder, on a plane, and on a sphere.

This transform is similar to a toroidal Radon transform, which maps a given function to its integrals over a set of tori. We also study this object.

1 Introduction

Photoacoustic Tomography (PAT) is a noninvasive medical imaging technique based on the reconstruction of an internal photoacoustic source. Its principle is based on the excitation of high bandwidth acoustic waves with pulsed non-ionizing electromagnetic energy [20, 23, 25]. Ultrasound imaging often has high resolution but displays low contrast. Optical or radio-frequency EM illumination, on the other hand, gives an enormous contrast between unhealthy and healthy tissues, although it has low resolution. The photoacoustic effect, which was discovered by A.G. Bell in 1880, is the underlying physical principle of PAT. PAT can provide information about the chemical composition as well as the optical structure of an object.

In PAT, one induces an acoustic wave inside an object of interest by delivering optical energy [12, 20], and then one measures the acoustic wave to a surface outside of the object of interest [5, 20, 23]. The initial data of the three dimensional wave equation contain diagnostic information. One of the mathematical problems of PAT boils down to recovering this initial pressure field.

The type of detector most often studied is a point transducer, which approximately measures the pressure at a given point. However, it is difficult to manufacture small detectors with high bandwidth and sensitivity. Hence, various other types of detectors to measure the acoustic data have been introduced, such as line detectors, planar detectors, cylindrical detectors and circular detectors. Measurements are modeled by the integrals of pressure over the shape of the detectors.

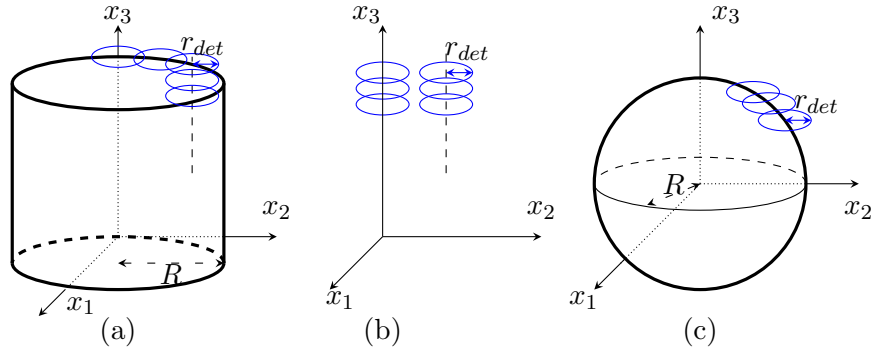


Figure 1: The centers of the circular detectors on (a) a cylinder, (b) a plane, and (c) a sphere

Works [23, 24, 25] have dealt with PAT with the circular detectors. They showed that the data from PAT with circular detectors is the solution of a certain initial value problem, and they converted the problem of recovering the initial pressure field into an inversion problem for the circular Radon transform using this fact. Also, they assume that the circular detectors are centered on a cylinder in [23, 25] or the circular detectors are of different radii and are lying on a surface of a sphere in [24]. In our approach, we define a new Radon-type transform arising in this version of PAT, and consider the situation when the set of the centers of the circular detectors is a cylinder (only this situation is discussed in previous works [23, 25]) and two more situations: when the set of the centers of the circular detectors is a plane or a sphere (this spherical geometry is different from that in [24]). This transform is similar to a toroidal Radon transform, which maps a given function into the set of its integrals over tori with respect to a certain non-standard measure; we also study this mathematically similar object.

This paper is organized as follows. Section 2 is devoted to a Radon-type transform arising in PAT with circular detectors. We reduce this Radon-type transform to the Radon transform on circles with a fixed radius. In section 3, we define a toroidal Radon transform and reduce this transform to the circular Radon transform.

2 PAT with circular integrating detectors

In PAT, the acoustic pressure $p(\mathbf{x}, t)$ satisfies the following initial value problem:

$$\begin{aligned} \partial_t^2 p(\mathbf{x}, t) &= \Delta_{\mathbf{x}} p(\mathbf{x}, t) & (\mathbf{x}, t) &= (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times (0, \infty), \\ p(\mathbf{x}, 0) &= f(\mathbf{x}) & \mathbf{x} &\in \mathbb{R}^3, \\ \partial_t p(\mathbf{x}, 0) &= 0 & \mathbf{x} &\in \mathbb{R}^3. \end{aligned} \tag{1}$$

(We assume that the sound speed is equal to one everywhere including the interior of object.) The goal of PAT is to recover the initial pressure f from measurements of p outside the support of f .

Throughout this section, it is assumed that the initial pressure field f is smooth and circular detectors are parallel to the x_1x_2 -plane. As mentioned before, three geometries will be studied. Let the centers of the circular detectors be located on a subset A of \mathbb{R}^3 . We consider three cases: A is the

cylinder $\partial B_R^2(0) \times \mathbb{R}$, the x_2x_3 -plane, or the sphere $\partial B_R^3(0)$ (see figure 1). Here $B_R^k(\mathbf{x}) = B^k(\mathbf{x}, R)$ is a ball in \mathbb{R}^k centered at $\mathbf{x} \in \mathbb{R}^k$ with radius R . In other words, it is assumed that the acoustic signals are measured by a stack of parallel circular detectors where these circles are centered on a cylinder $\partial B_R^2(0) \times \mathbb{R}$, on the x_2x_3 -plane, or on the sphere $\partial B_R^3(0)$ and their radii are a constant r_{det} .

The measured data $P(\mathbf{a}, t)$ for $(\mathbf{a}, t) \in A \times (0, \infty)$ can be written as

$$P(\mathbf{a}, t) = \frac{1}{2\pi} \int_0^{2\pi} p(\mathbf{a} + (r_{det}\vec{\alpha}, 0), t) d\alpha,$$

where $\vec{\alpha} = (\cos \alpha, \sin \alpha) \in S^1$. Also, it is a well-known fact that

$$p(\mathbf{x}, t) = \partial_t \left(\frac{1}{4\pi t} \int_{\partial B^3(\mathbf{x}, t)} f(\vec{\beta}) dS(\vec{\beta}) \right),$$

is a solution of the IVP (1). Here dS is the measure on the sphere and

$$\vec{\beta} = (\cos \beta_1 \sin \beta_2, \sin \beta_1 \sin \beta_2, \cos \beta_2) \in S^2.$$

Hence $P(\mathbf{a}, t)$ becomes

$$\begin{aligned} P(\mathbf{a}, t) &= \frac{1}{2\pi} \int_0^{2\pi} \partial_t \left(\frac{1}{4\pi t} \int_{\partial B^3(\mathbf{a} + (r_{det}\vec{\alpha}, 0), t)} f(\vec{\beta}) dS(\vec{\beta}) \right) d\alpha \\ &= \frac{1}{8\pi^2} \partial_t \left(t \int_0^{2\pi} \int_0^\pi \int_0^{2\pi} f(\mathbf{a} + (r_{det}\vec{\alpha}, 0) + t\vec{\beta}) \sin \beta_2 d\beta_1 d\beta_2 d\alpha \right). \end{aligned}$$

Let us define a transform \mathcal{R}_P by

$$\mathcal{R}_P f(\mathbf{a}, t) := \int_0^{2\pi} \int_0^{2\pi} \int_0^\pi f(\mathbf{a} + (r_{det}\vec{\alpha}, 0) + t\vec{\beta}) \sin \beta_2 d\beta_2 d\beta_1 d\alpha.$$

We will demonstrate a relation between the Radon transform on circles with a fixed radius—a well studied problem—and $\mathcal{R}_P f$. This will allow us to recover f from P .

2.1 Reconstruction

Let a transform $M_{r_{det}} f$ be defined by

$$M_{r_{det}} f(\mathbf{x}) := \int_0^{2\pi} f(r_{det}\vec{\alpha} + (x_1, x_2), x_3) d\alpha,$$

where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$.

We will show that $M_{r_{det}}f$ can be obtained from $\mathcal{R}_P f$ when A is a cylinder, a plane or a sphere.

We can easily find the inversion of the Radon transform $M_{r_{det}}f$ over all circles with a fixed radius as follows: Taking the 2-dimensional Fourier transform of $M_{r_{det}}f$ with respect to (x_1, x_2) , we have for $\boldsymbol{\xi} = (\xi_1, \xi_2) \in \mathbb{R}^2$,

$$\widehat{M_{r_{det}}f}(\boldsymbol{\xi}, x_3) = 2\pi \hat{f}(\boldsymbol{\xi}, x_3) J_0(r_{det}|\boldsymbol{\xi}|),$$

where $J_0(s)$ is the Bessel function of order zero, and $\widehat{M_{r_{det}}f}$ and \hat{f} are the 2-dimensional Fourier transforms of $M_{r_{det}}f$ and f with respect to (x_1, x_2) . Hence we can reconstruct f though

$$f(\mathbf{x}) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \widehat{M_{r_{det}}f}(\boldsymbol{\xi}, x_3) / J_0(r_{det}|\boldsymbol{\xi}|) e^{i\boldsymbol{\xi} \cdot (x_1, x_2)} d\boldsymbol{\xi}.$$

Remark 1. When we have two circular detectors with different radii, say r_1, r_2 , we have two different values $M_{r_1}f, M_{r_2}f$ for each \mathbf{x} , i.e., two Radon transforms on circles with different fixed radii. Some works [4, 19, 21] show how f can be reconstructed from $M_{r_1}f, M_{r_2}f$ under a certain assumption.

2.1.1 Cylindrical geometry

Let f be a smooth function supported in the solid cylinder $B_R^2(0) \times \mathbb{R} = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq R^2\}$. Let the centers of the circular detectors be located on the cylinder $A = \partial B^2(0) \times \mathbb{R}$. We can represent $\mathbf{a} \in A$ by $(R\vec{\theta}, z) \in \partial B_R^2(0) \times \mathbb{R}$ for $(\vec{\theta}, z) \in S^1 \times \mathbb{R}$. Then $\mathcal{R}_P f(R\vec{\theta}, z, t)$ is equal to

$$\int_0^{2\pi} \int_0^{2\pi} \int_0^\pi f((R\vec{\theta} + r_{det}\vec{\alpha}, z) + t\vec{\beta}) \sin \beta_2 d\beta_2 d\beta_1 d\alpha.$$

Consider the definition of $\mathcal{R}_P f$. The inner integral with respect to β_2 in the definition formula of $\mathcal{R}_P f$ can be thought of as the circular Radon transform with weight $\sin \beta_2$. We will first remove this integral by applying the technique previously used to derive an inversion formula for the circular Radon transform.

Let us define the operator $\mathcal{R}_P^\#$ for an integrable function g on $\partial B_R^2(0) \times \mathbb{R} \times [0, \infty)$ by

$$\mathcal{R}_P^\# g(R\vec{\theta}, x_3, \rho) = \int_{\mathbb{R}} g(R\vec{\theta}, z, \sqrt{(z - x_3)^2 + \rho^2}) |(z - x_3, \rho)| dz.$$

Lemma 2. Let $f \in C_c^\infty(B_R^2(0) \times \mathbb{R})$. Then we have

$$\int_0^{2\pi} \int_0^{2\pi} f(R\vec{\theta} + r_{det}\vec{\alpha} + t\vec{\beta}_1, x_3) d\beta_1 d\alpha = -\frac{1}{\pi^2 t} \mathcal{H}_t \partial_t \mathcal{R}_P^\# \mathcal{R}_P f(R\vec{\theta}, x_3, t). \quad (2)$$

Here $\mathcal{H}_t h$ is the Hilbert transform of h with respect to t defined by

$$\mathcal{H}_t h(t) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} h(\tau) \frac{d\tau}{t - \tau},$$

where P.V. means the principal value.

To prove this theorem, we follow a similar method to the one used in [1, 8, 15, 16].

Proof. By definition, $\mathcal{R}_P f(R\vec{\theta}, z, t)$ can be written as

$$-\int_0^{2\pi} \int_0^{2\pi} \int_{-1}^1 f(R\vec{\theta} + r_{det}\vec{\alpha} + t\sqrt{1-s^2}\vec{\beta}_1, z + ts) ds d\beta_1 d\alpha.$$

Taking the Fourier transform of $\mathcal{R}_P f$ with respect to z yields

$$\widehat{\mathcal{R}_P f}(R\vec{\theta}, \xi_1, t) = -\int_0^{2\pi} \int_0^{2\pi} \int_{-1}^1 \hat{f}(R\vec{\theta} + r_{det}\vec{\alpha} + t\sqrt{1-s^2}\vec{\beta}_1, \xi_1) e^{its\xi_1} ds d\beta_1 d\alpha,$$

where \hat{f} and $\widehat{\mathcal{R}_P f}$ are the 1-dimensional Fourier transforms of f and $\mathcal{R}_P f$ with respect to x_3 and z , respectively. Taking the Hankel transform of order zero of $t\widehat{\mathcal{R}_P f}$ with respect to t , we have

$$\begin{aligned} & H_0(t\widehat{\mathcal{R}_P f})(R\vec{\theta}, \xi_1, \eta) \\ &= -\int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \int_{-1}^1 \hat{f}(R\vec{\theta} + r_{det}\vec{\alpha} + t\sqrt{1-s^2}\vec{\beta}_1, \xi_1) e^{its\xi_1} ds d\beta_1 d\alpha t^2 J_0(t\eta) dt \\ &= -2 \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} \int_0^1 \hat{f}(R\vec{\theta} + r_{det}\vec{\alpha} + t\sqrt{1-s^2}\vec{\beta}_1, \xi_1) t^2 J_0(t\eta) \cos(ts\xi_1) ds d\beta_1 d\alpha dt \\ &= -2 \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^\infty \hat{f}(R\vec{\theta} + r_{det}\vec{\alpha} + b\vec{\beta}_1, \xi_1) b \cos(\rho\xi_1) J_0(\eta\sqrt{\rho^2 + b^2}) d\rho db d\beta_1 d\alpha, \end{aligned}$$

where in the last line, we made a change of variables $(t, s) \rightarrow (\rho, b)$ where $t = \sqrt{\rho^2 + b^2}$ and $s = \rho/\sqrt{\rho^2 + b^2}$. We use the following identity: for $0 < \xi_1 < a$,

$$\int_0^\infty J_0(a\sqrt{\rho^2 + b^2}) \cos(\rho\xi_1) d\rho = \begin{cases} \frac{\cos(b\sqrt{a^2 - \xi_1^2})}{\sqrt{a^2 - \xi_1^2}} & \text{if } 0 < \xi_1 < a, \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

(see [7, p.55 (35) vol.1]). Using this identity, $H_0(t\widehat{\mathcal{R}_P f})(R\vec{\theta}, \xi_1, \eta)$ is equal to

$$-2 \begin{cases} \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \hat{f}(R\vec{\theta} + r_{det}\vec{\alpha} + b\vec{\beta}_1, \xi_1) b \frac{\cos(b\sqrt{\eta^2 - \xi_1^2})}{\sqrt{\eta^2 - \xi_1^2}} db d\beta_1 d\alpha & \text{if } 0 < \xi_1 < \eta, \\ 0 & \text{otherwise.} \end{cases}$$

Substituting $\eta = \sqrt{\xi_1^2 + \xi_2^2}$ yields

$$H_0(t\widehat{\mathcal{R}_P f})(R\vec{\theta}, \xi_1, |\xi|) = -2 \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \hat{f}(R\vec{\theta} + r_{det}\vec{\alpha} + b\vec{\beta}_1, \xi_1) \frac{b}{\xi_2} \cos(b\xi_2) db d\beta_1 d\alpha. \quad (4)$$

The inner integral in the right hand side of the last equation is the Fourier cosine transform with respect to b , so taking the Fourier cosine transform of (4), we get

$$\begin{aligned} & \int_0^{2\pi} \int_0^{2\pi} \hat{f}(R\vec{\theta} + r_{det}\vec{\alpha} + s\vec{\beta}_1, \xi_1) s d\beta_1 d\alpha \\ &= -\pi^{-1} \int_0^\infty H_0(t\widehat{\mathcal{R}_P f})(R\vec{\theta}, \xi_1, |\xi|) \cos(s\xi_2) \xi_2 d\xi_2, \end{aligned} \quad (5)$$

where \hat{f} is the Fourier transform of f with respect to the last variable x_3 .

We change the right hand side of (5) into a term containing the operator $\mathcal{R}_P^\#$. Taking the Fourier transform of $\mathcal{R}_P^\# g$ on $\partial B^2(0) \times \mathbb{R}^2$ with respect to (z, ρ) yields

$$\begin{aligned} \widehat{\mathcal{R}_P^\# g}(R\vec{\theta}, \xi) &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(x_3, \rho) \cdot \xi} \mathcal{R}_P^\# g(R\vec{\theta}, x_3, \rho) dx_3 d\rho \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(x_3, \rho) \cdot \xi} \int_{\mathbb{R}} |(x_3 - z, \rho)| g(\theta, z, \sqrt{(z - x_3)^2 + \rho^2}) dz dx_3 d\rho \\ &= \int_{\mathbb{R}} e^{-i\xi_1 \cdot z} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(x_3 - z, \rho) \cdot \xi} |(x_3 - z, \rho)| g(\theta, z, \sqrt{(z - x_3)^2 + \rho^2}) dx_3 d\rho dz \\ &= \int_{\mathbb{R}} e^{-i\xi_1 \cdot z} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(x_3, \rho) \cdot \xi} |(x_3, \rho)| g(\theta, z, |(x_3, \rho)|) dx_3 d\rho dz \\ &= 2\pi \int_{\mathbb{R}} e^{-i\xi_1 \cdot z} H_0(tg)(R\vec{\theta}, z, |\xi|) dz \\ &= 2\pi H_0(t\hat{g})(R\vec{\theta}, \xi_1, |\xi|), \end{aligned} \quad (6)$$

where $\widehat{\mathcal{R}_P^\# g}$ is the Fourier transform with respect to the last variable (x_3, ρ) . Combining (6) with (5), we have for $g = \mathcal{R}_P f$,

$$\begin{aligned} \int_0^{2\pi} \int_0^{2\pi} \hat{f}(R\vec{\theta} + r_{det}\vec{\alpha} + s\vec{\beta}_1, \xi_1) d\beta_1 d\alpha &= -\frac{1}{2\pi^2 s} \int_0^\infty \widehat{\mathcal{R}_P^\# g}(R\vec{\theta}, \xi) \cos(s\xi_2) \xi_2 d\xi_2 \\ &= -\frac{1}{\pi^2 s} \int_0^\infty \widehat{\mathcal{R}_P^\# g}(R\vec{\theta}, \xi) e^{is\xi_2} |\xi_2| d\xi_2 \\ &= \frac{1}{\pi^2 s} \int_{\mathbb{R}} \widehat{\partial_t \mathcal{R}_P^\# g}(R\vec{\theta}, \xi) e^{is\xi_2} (i \operatorname{sgn}(\xi_2)) d\xi_2. \end{aligned}$$

The fact that $\widehat{\mathcal{H}_t h}(\xi) = (-i \operatorname{sgn}(\xi)) \hat{h}(\xi)$ completes our proof. \square

Again, the inner integral with respect to β_1 in the left hand side of (2) is the circular Radon transform with centers on $\partial B_R^2(0)$ and radius t . Hence, by applying an inversion formula of the circular Radon transform, we get $M_{r_{det}} f(\mathbf{x})$.

Theorem 3. Let f be a smooth function supported in $B_R^2(0) \times \mathbb{R}$. Then for any $\mathbf{x} \in \mathbb{R}^3$, we have

$$M_{r_{det}} f(\mathbf{x}) = \frac{1}{\pi R} \triangle_{x_1, x_2} \int_0^{2\pi} \mathcal{R}_P^\# \mathcal{R}_P f(R\vec{\theta}, x_3, |(x_1, x_2) - R\vec{\theta}|) d\theta.$$

To prove this theorem, we follow the method discussed in [9].

Proof. It is computed in [9] that

$$\begin{aligned} & \int_0^{2\pi} \log \left| |(x_1, x_2) - R\vec{\theta}|^2 - |(y_1, y_2) - R\vec{\theta}|^2 \right| d\theta \\ &= 2\pi R \log |(x_1, x_2) - (y_1, y_2)| + 2\pi R \log R. \end{aligned}$$

For any measurable function q on \mathbb{R} , it is easily shown that

$$\begin{aligned} & \int_0^{2(R+r_{det})} t \int_0^{2\pi} \int_0^{2\pi} f(R\vec{\theta} + r_{det}\vec{\alpha} + t\vec{\beta}_1, x_3) d\beta_1 d\alpha q(t) dt \\ &= \int_0^{2\pi} \int_{\mathbb{R}^2} f(R\vec{\theta} + r_{det}\vec{\alpha} + w, x_3) q(|\mathbf{w}|) d\mathbf{w} d\alpha. \end{aligned}$$

Applying this with $q(t) = \log \left| t^2 - |(x_1, x_2) - R\vec{\theta}|^2 \right|$ and making the change of variables $\mathbf{y} = (y_1, y_2) = R\vec{\theta} + t\vec{\beta}_1 \in \mathbb{R}^2$ give

$$\begin{aligned} & \int_0^{2\pi} \int_0^{2(R+r_{det})} \int_0^{2\pi} \int_0^{2\pi} t f(R\vec{\theta} + r_{det}\vec{\alpha} + t\vec{\beta}_1, x_3) \log \left| t^2 - |(x_1, x_2) - R\vec{\theta}|^2 \right| d\beta_1 d\alpha dt d\theta \\ &= \int_0^{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^2} f(r_{det}\vec{\alpha} + \mathbf{y}, x_3) \log \left| |(x_1, x_2) - R\vec{\theta}|^2 - |\mathbf{y} - R\vec{\theta}|^2 \right| d\mathbf{y} d\alpha d\theta \\ &= 2\pi R \int_0^{2\pi} \int_{\mathbb{R}^2} f(r_{det}\vec{\alpha} + \mathbf{y}, x_3) (\log |(x_1, x_2) - \mathbf{y}| + \log R) d\mathbf{y} d\alpha, \end{aligned}$$

where in the last line, we used the Fubini-Tonelli theorem. Since $(2\pi)^{-1} \log |(x_1, x_2) - (y_1, y_2)| + \log R$ is a fundamental solution of the Laplacian in \mathbb{R}^2 , we have

$$\begin{aligned} & \triangle \int_0^{2\pi} \int_0^{2(R+r_{det})} \int_0^{2\pi} \int_0^{2\pi} t f(R\vec{\theta} + r_{det}\vec{\alpha} + t\vec{\beta}_1, x_3) \log \left| t^2 - |(x_1, x_2) - R\vec{\theta}|^2 \right| d\beta_1 d\alpha dt d\theta \\ &= R \int_0^{2\pi} f((x_1, x_2) + r_{det}\vec{\alpha}, x_3) d\alpha, \end{aligned}$$

where Δ is the Laplacian on (x_1, x_2) . Applying Lemma 2, $M_{r_{det}}f(\mathbf{x})$ is equal to

$$-\frac{1}{\pi^2 R} \Delta_{x_1, x_2} \int_0^{2\pi} \int_0^{2(R+r_{det})} \mathcal{H}_t \partial_t \mathcal{R}_P^\# \mathcal{R}_P f(R\vec{\theta}, x_3, t) \log \left| t^2 - |(x_1, x_2) - R\vec{\theta}|^2 \right| dt d\theta.$$

We note that $\mathcal{H}_t \partial_t h = \partial_t \mathcal{H}_t h$,

$$\log \left| t^2 - |(x_1, x_2) - R\vec{\theta}|^2 \right| = \log \left| t - |(x_1, x_2) - R\vec{\theta}| \right| + \log \left| t + |(x_1, x_2) - R\vec{\theta}| \right|,$$

and $\log |t|$ is $P.V. \frac{1}{t}$. By integration by parts, we have

$$\begin{aligned} M_{r_{det}}f(\mathbf{x}) &= \frac{1}{\pi^2 R} \Delta_{x_1, x_2} \int_0^{2\pi} P.V. \int_0^{2(R+r_{det})} \frac{\mathcal{H}_t \mathcal{R}_P^\# \mathcal{R}_P f(R\vec{\theta}, x_3, t)}{t - |(x_1, x_2) - R\vec{\theta}|} dt d\theta \\ &\quad + \frac{1}{\pi^2 R} \Delta_{x_1, x_2} \int_0^{2\pi} \int_0^{2(R+r_{det})} \frac{\mathcal{H}_t \mathcal{R}_P^\# \mathcal{R}_P f(R\vec{\theta}, x_3, t)}{t + |(x_1, x_2) - R\vec{\theta}|} dt d\theta. \end{aligned}$$

Since $\mathcal{R}_P^\# \mathcal{R}_P f$ is even in t , so is $\mathcal{H}_t \mathcal{R}_P^\# \mathcal{R}_P f$. Substituting $t = -t$ in the second term gives

$$M_{r_{det}}f(\mathbf{x}) = \frac{1}{\pi^2 R} \Delta_{x_1, x_2} \int_0^{2\pi} P.V. \int_{-2(R+r_{det})}^{2(R+r_{det})} \frac{\mathcal{H}_t \mathcal{R}_P^\# \mathcal{R}_P f(R\vec{\theta}, x_3, t)}{t - |(x_1, x_2) - R\vec{\theta}|} dt d\theta.$$

The fact that $\mathcal{H}_t \mathcal{H}_t h(t) = -h(t)$ completes our proof. \square

Remark 4. When A is a cylinder, we can reconstruct f from $\mathcal{R}_P f$ by applying Theorem 3 and the argument below Remark 1.

2.1.2 Planar geometry

Let the centers of the circular detectors be located on the $x_2 x_3$ -plane. Then we can denote $\mathbf{a} \in A$ by $(y, z) \in \mathbb{R}^2$. Also, $\mathcal{R}_P f$ is equal to zero if f is an odd function in x_1 . We thus assume the function is even in x_1 . Then $\mathcal{R}_P f$ can be written by

$$\mathcal{R}_P f(y, z, t) = \int_0^{2\pi} \int_{S^2} f((0, y, z) + r_{det}(\vec{\alpha}, 0) + t\vec{\beta}) dS(\vec{\beta}) d\alpha.$$

Let M_P be the spherical Radon transform mapping a locally integrable function f on \mathbb{R}^3 into its integral over the set of spheres centered on the $x_2 x_3$ -plane:

$$M_P f(y, z, t) = \int_{S^2} f((0, y, z) + t\vec{\beta}) dS(\vec{\beta}).$$

Then we have

$$\mathcal{R}_P f(y, z, t) = \int_{S^2} M_{r_{det}} f((0, y, z) + t\vec{\beta}) dS(\vec{\beta}) = M_P(M_{r_{det}} f)(y, z, t).$$

It is well-known (see, e.g. [15, 16]) that for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$,

$$\hat{f}(\xi) = \frac{|\xi||\xi_1|}{4\pi^3} \mathcal{F}(M_P^* M_P f)(\xi),$$

where $\mathcal{F}f = \hat{f}$ is the 3-dimensional Fourier transform of f , and for an integrable function g on $\mathbb{R}^2 \times [0, \infty)$,

$$M_P^* g(\mathbf{x}) = \int_{\mathbb{R}^2} g(y, z, \sqrt{x_1^2 + (y - x_2)^2 + (z - x_3)^2}) dy dz.$$

Theorem 5. *Let $f \in C_c^\infty(\mathbb{R}^3)$ be even in x_1 . Then we have*

$$M_{r_{det}} f(\mathbf{x}) = \frac{1}{2^5 \pi^6} \int_{\mathbb{R}^3} |\xi||\xi_1| \mathcal{F}(M_P^* \mathcal{R}_P f)(\xi) e^{i\xi \cdot \mathbf{x}} d\xi.$$

Now f can be determined applying the argument below Remark 1.

Remark 6. *Redding and Newsam derived another inversion formula for the spherical Radon transform M_P in [18]. Using this inversion formula, we can also reconstruct $M_{r_{det}} f$ from $\mathcal{R}_P f$.*

2.1.3 Spherical geometry

Let the centers of the circular detectors be located on the sphere $\partial B_R^3(0)$. Then we can denote $\mathbf{a} \in A$ by $R\vec{\omega} \in \partial B_R^3(0)$ for $\vec{\omega} \in S^2$. Then $\mathcal{R}_P f$ can be written by

$$\mathcal{R}_P f(R\vec{\omega}, t) = \int_0^{2\pi} \int_{S^2} f(R\vec{\omega} + r_{det}(\vec{\alpha}, 0) + t\vec{\beta}) dS(\vec{\beta}) d\alpha.$$

Let M_S be the spherical Radon transform mapping a locally integrable function f on \mathbb{R}^3 into its integral over the spheres centered at the $\partial B_R^3(0)$:

$$M_S f(\vec{\omega}, t) = \int_{S^2} f(R\vec{\omega} + t\vec{\beta}) dS(\vec{\beta}).$$

Then we have

$$\mathcal{R}_P f(R\vec{\omega}, t) = \int_{S^2} M_{r_{det}} f(R\vec{\omega} + t\vec{\beta}) dS(\vec{\beta}) = M_S(M_{r_{det}} f)(\vec{\omega}, t).$$

It is well-known (see, e.g. [10]) that

$$f(\mathbf{x}) = -\frac{R}{2^3 \pi^2} \int_{S^2} \frac{\partial_t^2 t^2 M_S f(\vec{\omega}, t) \big|_{t=|R\vec{\omega}-\mathbf{x}|}}{|R\vec{\omega} - \mathbf{x}|} dS(\vec{\omega}).$$

Theorem 7. *Let $f \in C_c^\infty(B_R^3(0))$. Then we have*

$$M_{r_{det}}f(\mathbf{x}) = -\frac{R}{2^3\pi^2} \int_{S^2} \frac{\partial_t^2 t^2 \mathcal{R}_P f(R\vec{\omega}, t)|_{t=|R\vec{\omega}-\mathbf{x}|}}{|R\vec{\omega}-\mathbf{x}|} dS(\vec{\omega}).$$

Again f can be determined applying the argument below Remark 1.

Remark 8. *Kunyansky derived two other inversion formulas for the spherical Radon transform M_S in [13, 14]. Using these inversion formulas, we can reconstruct $M_{r_{det}}f$ from $\mathcal{R}_P f$.*

3 A toroidal Radon transform

As mentioned before, we study the toroidal Radon transform, which is a mathematically similar object to \mathcal{R}_P , in this section. (When integrating over the tori, the standard area measure is not used.) Although we have not been able to establish the direct link between PAT with circular detectors and the toroidal Radon transform, studying the toroidal Radon transform is an interesting geometric problem in its own right.

We assume that all tori are parallel to the x_1x_2 -plane and consider two geometries: the centers of tori are located on a cylinder, or on a plane.

Definition 9. *Let $u > 0$ be a radius of the central circles of tori. Let $A \times \mathbb{R} \subset \mathbb{R}^2 \times \mathbb{R}$ be the set of the centers of tori. The toroidal Radon transform R_T maps $f \in C_c^\infty(\mathbb{R}^3)$ into*

$$R_T f(\boldsymbol{\mu}, p, r) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} f(\boldsymbol{\mu} + (u - r \cos \beta)\vec{\alpha}, p + r \sin \beta) d\beta d\alpha, \quad (7)$$

for $(\boldsymbol{\mu}, p, r) \in A \times \mathbb{R} \times (0, \infty)$. Here α is the angular parameter along the central circle, $(\boldsymbol{\mu}, p)$ is the center of the torus, and β and r are the polar angle and the radius of the tube of the torus, respectively.

We consider two situations: A is the circle $\partial B_R^2(0)$ or the line $x_1 = 0$. Thus the set of the centers of tori is a cylinder $\partial B_R^2(0) \times \mathbb{R}$ or the x_2x_3 -plane. We then present the relation between the circular Radon transform and the toroidal Radon transform. This relation leads naturally to an inversion formula, if one uses an inversion formula for the circular Radon transform (already discussed in [9, 13] or [1, 8, 15, 16, 18]).

Definition 10. *Let f be a compactly supported function in \mathbb{R}^3 . The circular Radon transform M maps a function f into*

$$Mf(\boldsymbol{\mu}, x_3, r) = \int_0^{2\pi} f(\boldsymbol{\mu} + r\vec{\alpha}, x_3) d\alpha \quad \text{for } (\boldsymbol{\mu}, x_3, r) \in A \times \mathbb{R} \times (0, \infty).$$

3.1 Reconstruction

The inner integral with respect to β in (7) can be thought of as the circular Radon transform with radius r . As in subsection 2.1, we will first invert this transform.

Let us define the operator R_T^* for $g \in C_c^\infty(A \times \mathbb{R} \times [0, \infty))$ by

$$R_T^*g(\boldsymbol{\mu}, z, \rho) = \int_{\mathbb{R}} g(\boldsymbol{\mu}, p, \sqrt{(z-p)^2 + \rho^2}) dp,$$

where $(\boldsymbol{\mu}, z, \rho) \in A \times \mathbb{R}^2$. The following two lemmas show the relation between the circular and the toroidal Radon transforms.

Let us define the linear operator I_2^{-1} by $\widehat{I_2^{-1}h}(\boldsymbol{\mu}, \boldsymbol{\xi}) = |\xi_2| \hat{h}(\boldsymbol{\mu}, \boldsymbol{\xi})$, where h is a function on $A \times \mathbb{R}^2$ and \hat{h} is its Fourier transform in the last two-dimensional variable.

Lemma 11. *Let $f \in C_c^\infty(\mathbb{R}^3)$. Then we have*

$$\frac{1}{2} I_2^{-1} R_T^* R_T f(\boldsymbol{\mu}, x_3, r) = \begin{cases} Mf(\boldsymbol{\mu}, x_3, u-r) + Mf(\boldsymbol{\mu}, x_3, u+r) & \text{if } u > r, \\ Mf(\boldsymbol{\mu}, x_3, r-u) + Mf(\boldsymbol{\mu}, x_3, u+r) & \text{otherwise,} \end{cases} \quad (8)$$

To prove this lemma, we follow the method discussed in [1, 15, 16].

Proof. By definition, we have

$$R_T f(\boldsymbol{\mu}, p, r) = \frac{1}{2\pi} \sum_{j=1}^2 \int_0^{2\pi} \int_{-1}^1 f(\boldsymbol{\mu} + (u + (-1)^j r \sqrt{1-s^2}) \vec{\alpha}, p + rs) \frac{ds}{\sqrt{1-s^2}} d\alpha.$$

We take the Fourier transform of $R_T f$ with respect to p and the Hankel transform of order zero of $\widehat{R_T f}$ with respect to r . Then $H_0 \widehat{R_T f}(\boldsymbol{\mu}, \xi_1, \eta)$ can be written as

$$\frac{1}{2\pi} \sum_{j=1}^2 \int_0^{2\pi} \int_0^\infty \int_0^\infty \hat{f}(\boldsymbol{\mu} + (u + (-1)^j b) \vec{\alpha}, \xi_1) \cos(\rho \xi_1) J_0(\eta \sqrt{\rho^2 + b^2}) d\rho db d\alpha, \quad (9)$$

where \hat{f} and $\widehat{R_T f}$ are the 1-dimensional Fourier transforms of f and $R_T f$ with respect to z and p , respectively. Lastly, we change variables $(r, s) \rightarrow (\rho, b)$, where $r = \sqrt{\rho^2 + b^2}$ and $s = \rho/\sqrt{\rho^2 + b^2}$. Applying (3) to (9), we get

$$H_0 \widehat{R_T f}(\boldsymbol{\mu}, \xi_1, |\boldsymbol{\xi}|) = \frac{1}{2\pi} \sum_{j=1}^2 \int_0^{2\pi} \int_0^\infty \hat{f}(\boldsymbol{\mu} + (u + (-1)^j b) \vec{\alpha}, \xi_1) \frac{\cos(b \xi_2)}{\xi_2} db d\alpha.$$

The inner integral in the right hand side of the last equation is the Fourier cosine transform with respect to b , so taking the inverse Fourier cosine transform of the above formula, we get

$$\sum_{j=1}^2 \int_0^{2\pi} \hat{f}(\boldsymbol{\mu} + (u + (-1)^j s) \vec{\alpha}, \xi_1) d\alpha = 4 \int_0^\infty H_0 \widehat{R_T f}(\boldsymbol{\mu}, \xi_1, |\boldsymbol{\xi}|) \cos(s \xi_2) \xi_2 d\xi_2. \quad (10)$$

For a fixed ξ_1 , one recognizes the sum of two circular Radon transforms on the left.

Similarly to (6), we can change the right hand side of (10) into a term containing operator R_T^* , i.e.,

$$\widehat{R_T^*g}(\boldsymbol{\mu}, \boldsymbol{\xi}) = 2\pi H_0 \hat{g}(\boldsymbol{\mu}, \xi_1, |\boldsymbol{\xi}|). \quad (11)$$

Here $\widehat{R_T^*g}$ is the 2-dimensional Fourier transform with respect to the variables (z, ρ) . Combining (11) with (10), we have for $g = R_T f$,

$$\begin{aligned} \sum_{j=1}^2 \int_0^{2\pi} \hat{f}(\boldsymbol{\mu} + (u + (-1)^j s)\vec{\alpha}, \xi_1) d\alpha &= \frac{2}{\pi} \int_0^\infty \widehat{R_T^*g}(\boldsymbol{\mu}, \boldsymbol{\xi}) \cos(s\xi_2) \xi_2 d\xi_2 \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \widehat{R_T^*g}(\boldsymbol{\mu}, \boldsymbol{\xi}) e^{is\xi_2} |\xi_2| d\xi_2, \end{aligned}$$

since $\widehat{R_T^*g}$ is even in ξ_2 . □

Lemma 12. *Let $f \in C_c^\infty(\mathbb{R}^3)$. Then we have*

$$\begin{aligned} &\frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty r s R_T f(\boldsymbol{\mu}, -\eta, s) e^{-i(s^2 + 2x_3\eta + x_3^2 - \eta^2 + r^2)\xi} \xi ds d\eta d\xi \\ &= \begin{cases} Mf(\boldsymbol{\mu}, x_3, u - r) + Mf(\boldsymbol{\mu}, x_3, u + r) & \text{if } u > r, \\ Mf(\boldsymbol{\mu}, x_3, r - u) + Mf(\boldsymbol{\mu}, x_3, u + r) & \text{otherwise.} \end{cases} \end{aligned}$$

To prove this lemma, we follow the method discussed in [18].

Proof. Let G be defined by

$$G(\boldsymbol{\mu}, p, \xi) := \int_0^\infty r R_T f(\boldsymbol{\mu}, p, r) e^{-ir^2\xi} dr.$$

Then we have

$$\begin{aligned} G(\boldsymbol{\mu}, p, \xi) &= \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} \int_{-\pi}^\pi r f(\boldsymbol{\mu} + (u - r \cos \beta)\vec{\alpha}, p + r \sin \beta) e^{-ir^2\xi} d\beta d\alpha dr \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\boldsymbol{\mu} + (u - y)\vec{\alpha}, p + z) e^{-i(y^2 + z^2)\xi} dy dz d\alpha \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\boldsymbol{\mu} + (u - y)\vec{\alpha}, z) e^{-i(y^2 + (z-p)^2)\xi} dy dz d\alpha \\ &= \frac{e^{-ip^2\xi}}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\boldsymbol{\mu} + (u - y)\vec{\alpha}, z) e^{-i(y^2 + z^2)\xi} e^{2ipz\xi} dy dz d\alpha, \end{aligned}$$

where in the second line, we switched from the polar coordinates (r, β) to the Cartesian coordinates $(y, z) \in \mathbb{R}^2$. Making the change of variables $r = y^2 + z^2$ gives that $G(\boldsymbol{\mu}, p, \xi)$ equal

$$\frac{e^{-ip^2\xi}}{2\pi} \sum_{j=1}^2 \int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} f(\boldsymbol{\mu} + (u + (-1)^j \sqrt{r - z^2})\vec{\alpha}, z) \frac{e^{-ir\xi} e^{2ipz\xi}}{2\sqrt{r - z^2}} dr dz d\alpha.$$

Let us define the function

$$k_{\boldsymbol{\mu}}(\alpha, z, r) := \begin{cases} \sum_{j=1}^2 f(\boldsymbol{\mu} + (u + (-1)^j \sqrt{r - z^2})\vec{\alpha}, z) / \sqrt{r - z^2} & \text{if } 0 < z^2 < r, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} G(\boldsymbol{\mu}, p, \xi) &= \frac{e^{-ip^2\xi}}{4\pi} \int_0^{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} k_{\boldsymbol{\mu}}(\alpha, z, r) e^{-ir\xi} e^{2ipz\xi} dr dz d\alpha \\ &= \frac{e^{-ip^2\xi}}{4\pi} \int_0^{2\pi} \widehat{k_{\boldsymbol{\mu}}}(\alpha, -2p\xi, \xi) d\alpha, \end{aligned}$$

where $\widehat{k_{\boldsymbol{\mu}}}$ is the 2-dimensional Fourier transform of $k_{\boldsymbol{\mu}}$ with respect to the variables (z, r) . Also, we have

$$\begin{aligned} \sum_{j=1}^2 \int_0^{2\pi} f(\boldsymbol{\mu} + (u + (-1)^j s)\vec{\alpha}, x_3) d\alpha &= \int_0^{2\pi} s k_{\boldsymbol{\mu}}(\alpha, x_3, x_3^2 + s^2) d\alpha \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \int_0^{2\pi} s \widehat{k_{\boldsymbol{\mu}}}(\alpha, \eta, \xi) e^{-i(x_3\eta + (x_3^2 + s^2)\xi)} d\alpha d\eta d\xi \\ &= \frac{1}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} s e^{i\frac{\eta^2}{4\xi}} G(\boldsymbol{\mu}, -\frac{\eta}{2\xi}, \xi) e^{-i(x_3\eta + (x_3^2 + s^2)\xi)} d\eta d\xi \\ &= \frac{2}{\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} s G(\boldsymbol{\mu}, -\eta, \xi) e^{-i(2x_3\eta + (x_3^2 + s^2) - \eta^2)\xi} d\eta d\xi, \end{aligned}$$

where in the last line, we changed the variable $\eta/2\xi$ to η . □

3.1.1 Cylindrical geometry

Let the centers of the central circles be located on the a cylinder $\partial B_R^2(0) \times \mathbb{R} = A \times \mathbb{R}$. That is, A is the circle centered at the origin with radius R . The next two results show that the circular Radon transform can be recovered from the toroidal Radon transform. Both theorems are easily obtained using Lemma 11.

Theorem 13. *If $R/2 < u < R$ and $f \in C_c^\infty(B_{R-u}^2(0) \times \mathbb{R})$, then*

$$Mf(\boldsymbol{\mu}, x_3, r) = \begin{cases} 2^{-1} I_2^{-1} R_T^* R_T f(\boldsymbol{\mu}, x_3, r - u) & \text{if } r > u, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 14. Let $f \in C_c^\infty(B_R^2(0) \times \mathbb{R})$. Then

$$Mf(\boldsymbol{\mu}, x_3, r) = \begin{cases} \frac{1}{2} \sum_{j=0}^{\lfloor \frac{R+1}{u} \rfloor} (-1)^j I_2^{-1} R_T^* R_T f(\boldsymbol{\mu}, x_3, (2j+1)u - r) & \text{if } r \leq u, \\ \frac{1}{2} \sum_{j=0}^{\lfloor R/u \rfloor} (-1)^j I_2^{-1} R_T^* R_T f(\boldsymbol{\mu}, x_3, (2j+1)u + r) & \text{otherwise.} \end{cases}$$

Remark 15. One can obtain other relations similar to Theorems 13 and 14, by using Lemma 12 instead of Lemma 11.

Remark 16. When the set of centers of the circular detectors is a cylinder, (i.e., A is a circle,) one can recover f from its torodial transform $R_T f$ by applying inversion formulas for the spherical Radon transform with the centers of circles located on the circle (see e.g. [9, 13, 14]) to the left hand sides of equations in Theorems 13 and 14.

Remark 17. If $u > 2R$ (i.e., the radius of central circles is bigger than the diameter of the cylinder $B_R^2(0) \times \mathbb{R}$), then

$$Mf(\boldsymbol{\mu}, x_3, r) = \begin{cases} 2^{-1} I_2^{-1} R_T^* R_T f(\boldsymbol{\mu}, x_3, u - r) & \text{if } r \leq u, \\ 2^{-1} I_2^{-1} R_T^* R_T f(\boldsymbol{\mu}, x_3, u + r) & \text{otherwise.} \end{cases}$$

3.1.2 Planar geometry

Let $A \subset \mathbb{R}^2$ be the $x_1 = 0$ line (i.e., the centers of tori are located on the $x_2 x_3$ -plane in \mathbb{R}^3). Then $R_T f(\boldsymbol{\mu}, x_3, r)$ is equal to zero if f is an odd function in x_1 . We thus assume the function f to be even in x_1 .

Theorem 18. Let $f \in C_c^\infty(B_R^3(0))$ be even in x_1 . Then we have

$$Mf(\boldsymbol{\mu}, x_3, r) = \begin{cases} \frac{1}{2} \sum_{j=0}^{\lfloor \frac{R+u}{2u} \rfloor} (-1)^j I_2^{-1} R_T^* R_T f(\boldsymbol{\mu}, x_3, (2j+1)u - r) & \text{if } r \leq u, \\ \frac{1}{2} \sum_{j=0}^{\lfloor R/2u \rfloor} (-1)^j I_2^{-1} R_T^* R_T f(\boldsymbol{\mu}, x_3, (2j+1)u + r) & \text{otherwise.} \end{cases} \quad (12)$$

Remark 19. When A is a line, we can determine f from $R_T f$ by applying inversion formulas for the spherical Radon transform with the centers of circles on the hyperplane [1, 8, 15, 16, 18] to the left hand side of (12).

Remark 20. If $u > R$ (i.e., the radius of the detectors is bigger than the radius of the ball containing $\text{supp } f$), then

$$Mf(\boldsymbol{\mu}, x_3, r) = \begin{cases} 2^{-1} I_2^{-1} R_T^* R_T f(\boldsymbol{\mu}, x_3, u - r) & \text{if } r < u, \\ 2^{-1} I_2^{-1} R_T^* R_T f(\boldsymbol{\mu}, x_3, u + r) & \text{otherwise.} \end{cases}$$

4 Conclusion

Here we studied a Radon-type transform arising in PAT with circular detectors, and also the toroidal Radon transform. We proved that these transforms reduce to well-studied transforms: the Radon transform over circles with a fixed radius, or the circular Radon transform.

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